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AUTOMORPHISMS OF FREE NONASSOCIATIVE ALGEBRAS OVER EUCLIDEAN RINGS

ABSTRACT

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It is known that automorphisms describe all the symmetries of an algebraic object and related geometric objects. Therefore, the study of automorphisms of free algebras is important, since it helps to determine the situation in the structures of these algebras, to classify and systematize them. Many mathematicians have studied the automorphism groups of free groups and algebras.

In 1924, J. Nielsen proved a number of deep theorems on the structure of groups of automorphisms of free groups. In particular, he obtained a finite representation of the automorphism group of a free finitely generated group in terms of generators and defining relations. The generating elements of the group of automorphisms of free finitely generated groups later came to be called elementary. Elementary automorphisms have also been defined for many other free groups and algebras. Automorphisms of a free group or algebra are called tame if they are generated by elementary automorphisms, and the remaining automorphisms are called wild.

In 1942, H.W.E. Jung proved that automorphisms of the algebra of polynomials $K[x, y]$ in two variables over a field K of characteristic zero are tame. In 1953, W. van der Kalk generalized this result for fields of an arbitrary characteristic. Moreover, the automorphism group $\mathrm{Aut}(K[x, y])$ of this algebra admits the structure of an amalgamated free product, that is,

$$\mathrm{Aut}(K[x, y]) = A *_C B,$$

where A is the subgroup of affine automorphisms, B is the subgroup of triangular automorphisms, and $C = A \cap B$. This result gives an exhaustive description of the automorphism groups of two-generated polynomial algebras and was actually proved by W. van der Kalk. Although for the first time the exact formulation was given by I.R. Shafarevich in 1966. In D. Wright's work an analogue of this result was proved for tame automorphisms of two-generated polynomial algebras over an arbitrary integrity domain. Similar results also hold for free associative algebras and for free Poisson algebras over a field of characteristic zero, and the groups of automorphisms of these algebras are isomorphic to the group of automorphisms of the of polynomial algebras.

L. Makar-Limanov, D. Kozybaev, U. Umirbaev proved that automorphisms of free right-symmetric algebras of rank two are tame. The automorphism groups of

these algebras are much broader than the automorphism groups of polynomial algebras due to the rich structure of elementary automorphisms.

In 1964, P. Cohn proved that automorphisms of finitely generated free Lie algebras over an arbitrary field are tame. J. Levin generalized this result for Schreier varieties of algebras. Recall that Schreier varieties are varieties of all nonassociative algebras, commutative and anticommutative algebras, Lie algebras, and Lie superalgebras. Consequently, automorphisms of free nonassociative algebras, free commutative and anticommutative algebras of finite rank over an arbitrary field are also tame.

In 1979, using the structure of an amalgamated free product of the automorphism group of $\text{Aut}(K[x, y])$ T. Kambayashi proved that any algebraic subgroup of the group of automorphisms $\text{Aut}(K[x, y])$ is conjugate with a subgroup of linear or triangular automorphisms. It follows from this that the action of any reductive group on K^n, where $n = 2$, is linearizable. T. Kambayashi also assumed that this result is valid for all $n > 2$. This assumption is called the linearization hypothesis for actions of reductive groups. It turned out that this hypothesis is not true for $n \geq 4$. In 1989, Schwartz constructed the first counterexamples of non-linearizable actions of reductive groups. Later, examples of non-linearizable actions of finite groups were constructed.

In 1968, R. Renchler proved that locally nilpotent derivations of the algebra of polynomials in two variables over a field of characteristic zero are triangulable. H. Bass constructed an example of a non-triangulable derivation of an algebra of polynomials in three variables.

In this dissertation, it is proved that the group of automorphisms of a free right-symmetric algebra of rank two admits the structure of an amalgamated free product. Using this structure, the linearizability of the reductive automorphism group and the triangulability of locally nilpotent differentiations of a free right-symmetric algebra of rank two in the case of characteristic zero are also proved. Similar results are also true for free nonassociative algebras and free commutative algebras of rank two. Note that the automorphisms of free Lie algebras and free anticommutative algebras of two variables are linear.

The structure of the amalgamated free product is used for the first time in this dissertation to prove the triangulability of locally nilpotent differentiations.

In 1972, M. Nagata constructed an automorphism

$$\sigma = (x + 2yw + w^2 z, y + wz, z), w = xz - y^2$$

and he hypothesized that this automorphism is not tame (i.e. it is wild). In 2004, I. Shestakov and U. Umirbaev proved that the Nagata automorphism σ is wild in the case of fields of characteristic zero. Nagata's automorphism gives an example of wild automorphisms of free Poisson algebras of three variables.

In the case of free associative algebras of rank three, the question of the existence of wild automorphisms (Kohn's problem) was solved by U.U. Umirbaev. He proved that the automorphism of Anik

$$\delta = (x + z(xz - zy), y + (xz - zy)z, z)$$

a free associative algebra $K\langle x, y, z\rangle$ over a field of characteristic zero is wild.

In this dissertation work, an example of a wild automorphism is constructed

$$\eta = (x_1 + x_2(zx_1 - x_2^2) + (zx_1 - x_2^2)x_2 + z(zx_1 - x_2^2)^2, x_2 + z(zx_1 - x_2^2))$$

a free nonassociative algebra and a free commutative algebra of rank two over an arbitrary Euclidean ring. This automorphism is a generalization of the Nagata automorphism and induces the Nagata automorphism of the algebra $K[x, y, z]$. Nagata proved that σ is a wild automorphism of the algebra $K[z][x, y]$ over $K[z]$. For this reason, the automorphism η induces a wild automorphism of free algebras of any variety of algebras containing the polynomial algebras. In the dissertation, another class of varieties of algebras is defined for which η is a wild automorphism. Note that the method of constructing the automorphism η does not work for anticommutative algebras, since the square of the element in anticommutative algebras is zero. Generalization of methods for constructing automorphisms of Nagata and Anika allowed in this paper to construct a wild automorphism

$$\delta = (x_1 + [zx_1 - [x_2, x_3], x_3], x_2 + z(zx_1 - [x_2, x_3]), x_3)$$

Lie algebras of rank 3 over an arbitrary Euclidean ring. This automorphism is analogous to Anik's automorphism of free associative algebras.

The aim of the work is to research automorphisms and differentiations of two-generated free nonassociative algebras and three-generated free Lie algebras over Euclidean rings.

The object of the research is automorphisms and differentiations of two-generated free nonassociative algebras and three-generated free Lie algebras over Euclidean rings.

Research methods. The paper uses the method of automorphism reduction, methods and results of group theory, structural and combinatorial theory of nonassociative algebras and Lie algebras.

The main provisions submitted for defense and the results of the study. The main results of the dissertation research are as follows

– an example of a wild automorphism of a free nonassociative algebra and a free commutative algebra of rank two over an arbitrary Euclidean ring is constructed;

– a wide class of ∗-varieties of algebras is defined and it is proved that the group of tame automorphisms of free two-generated algebras of any ∗-variety of algebras over an arbitrary integrity domain admits the structure of an amalgamated free product;

– the contractility of tame automorphisms of free algebras in two variables is proved for ∗ -varieties of algebras over an arbitrary Euclidean ring;

– the algorithmic recognizability of tame automorphisms of a two-generated free algebra of ∗ -varieties of algebras over a constructive Euclidean ring is proved;

– an example of a wild automorphism of a free algebra of rank two of any * -varieties of algebras over an arbitrary Euclidean ring is constructed;

– the class of ∘-varieties of algebras is defined, which is a subclass of the class of * -varieties of algebras, and it is proved that the variety of right-symmetric algebras is also ∘-variety;

– the linearizability of the reductive group of automorphisms of a free algebra in two variables of any ∘-variety of algebras over an arbitrary field of characteristic zero is proved;

– the triangulability of locally nilpotent differentiations of free algebras of rank two of any ∘-variety of algebras over a field of characteristic zero is proved;

– as a consequence of the last two results, the linearizability of the reductive automorphism group and the triangulability of locally nilpotent differentiations of free associative, free right-symmetric, free nonassociative and free commutative algebras of rank two over a field of characteristic zero are proved;

– it was proved that the group of tame automorphisms of a free Lie algebra (and a free anticommutative algebra) in three variables over an arbitrary integrity domain admits the structure of an amalgamated free product;

– it was proved that the group of automorphisms of a free Lie algebra in three variables over an arbitrary field can be represented as an amalgamated free product;

– the cancellation of tame automorphisms of a free Lie algebra in three variables over an arbitrary Euclidean ring is proved;

– Proved algorithmic recognizability of tame automorphisms of a free Lie algebra in three variables over a constructive Euclidean ring;

– an example of a wild automorphism of a free Lie algebra (and a free anticommutative algebra) of rank 3 over a Euclidean ring is constructed.

Scientific novelty of the research. All the main results of the dissertation are new.

Theoretical and practical significance. The work is theoretical in nature. The methods used and the results obtained can be applied to further study of free nonassociative algebras and free Lie algebras over Euclidean rings. And also the results of this work can be included in special courses on the theory of free nonassociative algebras and their automorphisms.

Approbation of the results. The main results of the dissertation were reported:

– at the algebraic seminar of the Department of Algebra and Geometry of the Faculty of Mechanics and Mathematics of the L.N. Gumilev Eurasian National University (Nur-Sultan, 2015-2020);

– at the international conference "Lie and Jordan algebra, their representations and applications-VI, dedicated to Efim Zelmanov's 60th birthday", (Bento Goncalves, Brazil, December 13-19, 2015);

– at the algebraic seminar of the Faculty of Mathematics of Wayne State University (Detroit, USA, 2015);

– at the VI International Congress of the Mathematical Society of the Turkic World (Astana, October 2-5, 2017);

- at the International conference on Algebra and Mathematical Logic "Maltsev Readings" (Novosibirsk, Russia, November 19-22, 2018).

Publications. The results of the dissertation have been published in 7 papers, including 1 article in the journal included in the 3rd quartile (Q3) in the Web of Science database, 2 articles in the journal with a percentile of 32 CiteScore in the Scopus database, 1 article in the journal from the list recommended by the Committee for Control in the Field of Education and Science of the Ministry of Education and Science of the RK, 3 papers in the materials of international conferences.

Structure and scope of work. The dissertation contains 71 pages, consists of an introduction, three sections, a conclusion and a list of sources used, which includes 49 titles.